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Transcribing spacetime data into matrices

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Abstract: In certain supergravity backgrounds, D0 branes may polarize into higher dimensional Dp branes. We study this phenomenon in some generality from the perspective of a local inertial observer and explore polarization effects resulting from tidal-like forces. We find D2 brane droplets made of D0 branes at an extremum of the Born-Infeld action even in scenarios where the RR fields may be zero. These solutions lead us to a local formulation of the UV-IR correspondence. A holographic Planck scale bound on the number of D0 branes plays an important role in the analysis. We focus on the impact of higher order moments of background fields and work out extensions of the non-commutative algebra beyond the Lie and Heisenberg structures. In this context, it appears that q-deformed algebras come into play.

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1. Introduction

Recently, a deeper understanding of properties of D branes in supergravity backgrounds was achieved in the work of [1, 2]. A host of new interaction terms were identified in the Dirac-Born-Infeld (DBI) action that explore the non-commutative dynamics of the D-brane coordinates. In particular, it was pointed out that RR fields carrying the charges of Dp branes, along with momentum modes in certain special scenarios, can polarize D0 branes [1], and [3]–[6]. The resulting objects appear to be microscopic realizations of higher dimensional Dp branes. Such configurations have typically non-commutative field theories living on their worldvolumes [7, 8, 9], and encode in their shape information about the background space. D-branes are natural probes of the Planck scale structure of space-time [10]. They entail exotic dynamics, involving stringy non-local interactions, and are non-perturbative in character [11]. It then becomes important to understand how data in the spacetime fields gets transcribed through polarization into the matrices representing the coordinates of D branes.

In this work, we investigate couplings of D0 branes to most of the supergravity bosonic fields of the IIA theory from the perspective of a local inertial observer. In addition to polarization effects from RR gauge fields, tidal-like forces arising from

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various interaction terms in the DBI action can also polarize the D0 branes. We find ellipsoidal droplets of D0 branes that store some of the information about the supergravity background in their shape. Furthermore, in certain regions of the background field parameter space where we may naively expect polarization by tidal forces, these ellipsoids are unstable. There is a game of competition between couplings of the D branes to the supergravity fields and the forces binding the N branes together through strings stretched between them. By looking for such configurations at the extrema of the DBI action, we effectively probe into the distribution of these forces within the polarized ellipsoid.

Throughout our discussion, we will be ignoring effects of back-reaction from the D0 branes onto the background spacetime. Within this approximation scheme, we find that the regime of validity of the expansion of the DBI action arises as a statement bounding the number of D0 branes by a local measure of area in Planck units. Given a characteristic length scale L for background field variations (relating to, for example, the local scale for the spacetime curvature), N D0 branes living in a three dimensional subset of the transverse space must satisfy the bound $N \ll L^2/\ell_{\rm pl}^2$. A second observation is a local realization of the UV-IR correspondence [12]. We argue that the scale of non-commutativity in the worldvolume theory of the polarized D0 brane configuration is inversely proportional to the length scale characterizing local variations in the background fields.

To store information about higher moments of background fields into D0 brane configurations, one needs to go beyond the Lie algebraic structure. We present a prescription on how to encode this additional data in generalized algebras. We find that, at the next order in the expansion, the spacetime gets transcribed into algebras with q-deformed structure [13, 14].

We note that the phenomenon we investigate has to do with polarization effects involving a large number of D0 branes. Recently, other authors have explored somewhat related dynamics arising from the fermionic degrees of freedom on a single D0 brane [15, 16].

In section 2, we setup the actions and equations of motion of interest; we present several simple solutions and analyze the underlying dynamics. In section 3, we consider effects higher order in the string tension, and describe extensions of Lie algebras that solve the equations of motion. In section 4, we briefly outline a non-compact solution describing a non-commutative hyperboloid that can be realized with infinite size matrices. The Appendix contains a few technical details used in the main text.

Note added: The solutions we find are at an extremum of the energy and there is an issue of stability that needs to be addressed. I am grateful to M. Spradlin and A. Volovich for bringing this issue to my attention. We comment on this problem briefly in the text, and defer a detailed analysis to an upcoming work [38].

2. Polarization with Lie algebraic structure

In this section, we study static solutions with U(2) algebraic structure describing N D0 branes in background supergravity fields. In the first subsection, we set up the action and the equations of motion to cubic order in the inverse string tension. In section 2.2, we write solutions for backgrounds where all fields but the D2 brane gauge field are nonzero. In section 2.3, we address some issues regarding the stability of these configurations and the regime of validity of our classical calculation, formulating a local statement regarding the UV-IR correspondence. We end with section 2.4 by considering the effect of the D2 brane gauge field on the dynamics. Some details of this section are sketched in the Appendix.

2.1 The setup and equations of motion

Consider N D0 branes emersed in a general type IIA supergravity background. The branes are described by $N \times N$ hermitian matrices Φ^i, with $i = 1..9$. The dynamics in the energy regime of interest is governed by the non-abelian DBI action [1]

$$S = -\frac{1}{g_{\rm str}\ell_{\rm str}}\int dt\ \mathrm{STr}\left\{e^{-\phi}\left(-\left(P\left[E_{00}+E_{0i}\left(Q^{-1}-\delta\right)^{ij}E_{j0}\right]\right)\right)^{1/2}(\det Q)^{1/2}\right\}+ \\ +\frac{1}{g_{\rm str}\ell_{\rm str}}\int \mathrm{STr}\left\{P\left[e^{i\lambda i_\Phi i_\Phi}\left(\sum C^{(n)}e^{B}\right)\right]\right\}\ , \tag{2.1}$$

where the matrix Q is defined by

$$Q^i_j \equiv \delta^i_j + i\lambda\left[\Phi^i,\Phi^k\right]E_{kj}\ , \tag{2.2}$$

and

$$E_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}\ . \tag{2.3}$$

We will ignore the back reaction effects of the D0 branes on the background. Throughout, we will follow closely the notation introduced in [1]. The space-time metric and NS-NS gauge field are denoted by $G_{\mu\nu}$ and $B_{\mu\nu}$ respectively; we also have the dilaton field ϕ and the RR gauge fields $C^{(n)}$; $\lambda \equiv 2\pi\ell_{\rm str}^2$ is the inverse string tension, and $g_{\rm str}$ is the IIA string coupling. The Φ^i's appear also in equation (2.1) implicitly through the dependence of the supergravity fields on the spacetime coordinates [17, 18, 1]

$$\psi \equiv e^{\lambda\Phi^i\partial_i}\psi(x)\Big|_P\ , \tag{2.4}$$

where P is a point about which we expand the fields, and ψ represents any of the supergravity fields; this corresponds to a standard normal ordering prescription. Φ^i's are also hidden in the pull-back of the fields to the world volume of the D0 branes, denoted by $P[\psi_0] \equiv \psi_0 + \lambda\dot{\Phi}^i\psi_i$ in the canonical static gauge, where the dot denotes differentiation with respect to time. Finally, i_Φ denotes the interior product operator, and STr is short-hand for the symmetrized trace prescription first introduced in [19].

In the cases of interest, it will be useful to rescale the metric so as to eliminate the dilaton field appearing in front of the DBI action. This trick will allow us to consider geodesic motion that incorporates the effects of the coupling of the D0 branes to the dilaton field.[1] Consider the scenario where we let go of a number of D0 branes in a generic background field configuration, and observe the center of mass of the D0 branes as it follows a geodesic in the rescaled metric. The viewpoint of the freely falling D0 branes is captured by choosing Fermi normal coordinates [20]

$$G_{\mu\nu}|_P = \eta_{\mu\nu}\,, \qquad G_{\mu\nu,\alpha}|_P = 0\,, \tag{2.5}$$

where P is a point along the geodesic. It is known that the metric seen in this reference frame will vary in time adiabatically, so that one can consider static solutions at an instant, and evolve them in time trivially. We also assume that all other supergravity fields vary slowly in the observer's time coordinate as well. This can sometimes be arranged by a judicious choice of the initial conditions for the geodesic motion. We then drop all time derivatives of all fields, and look for static configurations of N D0 branes in this freely falling frame with

$$\dot{\Phi}^i = 0\,. \tag{2.6}$$

Using constant gauge transformations, we can set the value of any gauge field at point P equal to any constant value; for example, we choose

$$B_{\mu\nu}|_P = 0\,. \tag{2.7}$$

Furthermore, the coupling $g_{\rm str}$ in front of (2.1) is taken as the value of e^{ϕ} at point P and is not used in the rescaling of the metric. We then expand (2.1) in powers of λ. We will quantify the regime of validity of this expansion more carefully later. As a statement relevant only to terms of order λ^3 and beyond, we require additionally that[2]

$$B_{ti,j} = 0\,. \tag{2.8}$$

The main technical simplifications that results in this game are that the term $(Q^{-1} - \delta)$ in (2.1) does not contribute to order λ^3, and that the pull-back is trivial.

The strategy is to expand the DBI action while dropping all terms involving only matrices proportional to the identity. This is because these terms describe the dynamical evolution of the center of mass of the D0 branes, which we will solve for separately using the full DBI action without an expansion. Stated differently, the

[1] In a previous version of this paper, this approach was not used and the effect of the dilaton field appeared explicitely in subsequent equations. The current approach is clearer and simplifies the discussion.

[2] Part of the motivation for this statement is that the effect of the NS-NS electric field on D branes may be better explored with D-instantons probes. More on this issue in the Discussion section.

terms proportional to the identity will be summed back to the square root and the U(1) part of the ansatz for the Φ^i's will involve time dependence. The rest of the problem involves looking for static solutions in SU(N), with the U(1) factored out. This procedure assumes that the size of the D0 branes in the center of mass frame will not affect its center of mass trajectory. This would be the case if the size of the polarized D0 branes is much smaller than the typical wavelength over which the background fields vary; so that the center of mass dynamics is that of a point-like object. To order λ^2, this decoupling of the U(1) and SU(N) sectors in the dynamics can be easily seen. To order λ^3, this is more subtle.[3]

Consider first an expansion to order λ^2; the form of the action can be determined (almost uniquely) by noting that each Φ^i arising in a symmetric combination and each commutator of the Φ^i's come with a power of λ, and by making use of the symmetrized trace prescription. We are led to the structure[4]

$$S = -\frac{1}{g_{\rm str}\ell_{\rm str}} \int dt\ \mathrm{STr}\Big\{ +\frac{\lambda^2}{2} M_{ij} \left\{\Phi^i, \Phi^j\right\} + i\lambda^2 N_{ikl} \left[\Phi^i, \Phi^k\right] \Phi^l + \\ + \lambda^2 P_{ijkl} \left[\Phi^l, \Phi^k\right] \left[\Phi^j, \Phi^i\right] + O(\lambda^3) \Big\}\,, \tag{2.9}$$

where M_{ij}, N_{ijk} and P_{ijkl} are c-number background fields evaluated at point P. These fields then acquire the following properties

$$\begin{aligned} &M_{ij} = M_{ji}\,, \qquad N_{ijk} = N_{[ijk]}\,; \\ &P_{ijkl} = -P_{jikl}\,, \qquad P_{ijkl} = -P_{ijlk}\,, \qquad P_{[ijk]l} = 0 \quad \Rightarrow \quad P_{ijkl} = P_{klij}\,. \end{aligned} \tag{2.10}$$

Hence, the P field has the properties of a Riemann curvature tensor; not surprisingly, since the commutator structure multiplying it in the action is related to a curvature form on a Lie algebra defined by the Φ^i's.

Expanding (2.1), we obtain the following realizations for the background fields

$$M_{ij} \equiv -\frac{1}{4} G_{00,ij} - \frac{1}{2} C^{(1)}_{0,ij}\,; \tag{2.11}$$

$$N_{ikl} \equiv \frac{1}{2} B_{[ki,l]} - \frac{1}{2} C^{(3)}_{0[ki,l]} - \frac{1}{2} C^{(1)}_0 B_{[ki,l]}\,; \tag{2.12}$$

$$P_{ijkl} \equiv \frac{1}{4}\delta_{kj}\delta_{li} + \frac{1}{8} C^{(5)}_{ijkl0} \to \frac{1}{4}\delta_{kj}\delta_{li}\,. \tag{2.13}$$

It is important to emphasize that all supergravity fields appearing in these equations are evaluated at the point P.[5] We also have made use of the choices for coordinates

[3] In a previous version of this work, this decoupling issue was addressed explicitely, culminating in finding (redundantly) unstable modes in the U(1) sector. In our current approach, this issue gets circumvented and the treatment is more transparent.

[4] We note that the Φ^i's have dimension of inverse length; i.e. the combination $\lambda\Phi^i$ relates to spacetime coordinates.

[5] Note that the value of the dilaton at point P is factored out in front of the action and the rest of the dilaton field was absorbed in the metric, denoted here as $G_{\mu\nu}$.

and gauges described above. In the last equation, the D4 brane gauge field does not contribute due to the symmetry relations (2.10).[6]

The equations of motion that follow from (2.9) are

$$2\lambda M_{in}\Phi^i + 3i\lambda N_{nkl}\left[\Phi^k,\Phi^l\right] + \lambda\left[\Phi^j,\left[\Phi^j,\Phi^n\right]\right] = 0\,. \tag{2.14}$$

In a second scenario of interest, we would like to study the effect of the couplings appearing in (2.1) at order λ^3. To avoid overwhelming ourselves with too much new physics, we will set all fields to zero, except the metric and the dilaton. Expanding (2.1) to order λ^3, we get

$$S = -\frac{1}{g_{\rm str}\ell_{\rm str}}\int dt\ \ \mathrm{STr}\left\{\frac{\lambda^2}{2}M_{ij}\left\{\Phi^i,\Phi^j\right\} + \frac{\lambda^2}{4}\left[\Phi^i,\Phi^j\right]\left[\Phi^j,\Phi^i\right] + \right.$$
$$\left. + \lambda^3\Phi^i\Phi^j\Phi^k T_{ijk} + O(\lambda^4)\right\}. \tag{2.15}$$

We have defined

$$T_{ijk} \equiv -\frac{1}{12}G_{00,ijk}\,. \tag{2.16}$$

The equations of motion become

$$2\lambda M_{in}\Phi^i + \lambda\left[\Phi^j,\left[\Phi^j,\Phi^n\right]\right] + \frac{3}{2}\lambda^2\left\{\Phi^i,\Phi^j\right\}T_{nij} = 0\,. \tag{2.17}$$

Looking for solutions to (2.14) and (2.17), we focus on backgrounds which are non-trivial only in a three dimensional subset of the full nine dimensional space; i.e. there are no background fields turned on with components in the other six space dimensions, and all fields have only dependence on coordinates within the chosen three dimensional subset. This means that the indices $i, j, k, \ldots$ run from one to three. We then have three matrices Φ^i appearing in the action for which we will need to solve for, while the rest are set to zero. Physically, this means that we have arranged for a situation where the D0 branes sit on top of each other in six of the nine dimensions of the transverse space, and acquire non-trivial configurations in the remaining three. Later on, we will briefly comment on generalizations of this setup to situations that explore larger dimensional background spaces.

In section 2.2, we will look for solutions to (2.14) with Lie algebraic structure. The relevant algebra is SU(2), and deformations and contractions of it. The matrices are in an $N \times N$ representation. The case where $M_{ij} = 0$ was discussed in [1]. To order λ^2, new physics will arise from the term M_{ij}, which encodes, partly, the effects of polarizing the N D0 branes by the background curvature of space, i.e. by the tidal forces in the local inertial frame.

[6]It is amusing to note that, in the three dimensional setting we will be focusing on, the "Weyl tensor" associated with the P field vanishes; hence, the content of this field in general is that of a symmetric two tensor. Yet, we have checked that, in looking for solutions to our equations, the only "physically non-trivial" content of P comes about from the term shown in (2.13).

In section 3, we will study solutions of (2.17) and find that Lie algebraic structure is insufficient to encode all of the data in the background fields into matrices; the algebra that solves these equations will acquire structure similar to ones that arise in the context of q-deformed algebras.

2.2 The non-commutative ellipsoid

We consider solutions of (2.14) in backgrounds confined to a three dimensional subset of the full nine dimensional transverse space, as described in the previous section. The field N_{ijk} is then given by

$$N_{ijk} = \tilde{N}\varepsilon_{ijk}\,, \tag{2.18}$$

where

$$\tilde{N} = \frac{1}{2}\left(C^{(3)}_{012,3} + H_{123}\left(C^{(1)}_{0} - 1\right)\right)\,. \tag{2.19}$$

We look for configurations of matrices Φ^i in SU(2)

$$\Phi^i = \sigma^i\,, \tag{2.20}$$

that solve (2.14), with the σ^i's obeying

$$\left[\sigma^i, \sigma^j\right] = C^{ij}{}_k\sigma^k\,. \tag{2.21}$$

The unknowns are the structure constants and we write them as[7]

$$C^{ij}{}_k = i\ \varepsilon^{ijl}\ g_{lk}\,, \tag{2.22}$$

where the metric g_{lk} is proportional to the Cartan-Killing metric.[8] Equation (2.14) determines this metric; i.e. we are looking for the particular linear combination of the canonical SU(2) matrices that solve the equation. This leads us to

$$2M_{ln} - 6\tilde{N}g_{nl} - g_{lr}g_{rn} + g_{ii}g_{nl} = 0\,. \tag{2.23}$$

Let us first look at the case where $M_{ij} = 0$. Then equation (2.23) is linear in g_{ij} and the solution is simply

$$C^{kn}{}_m = 3i\tilde{N}\varepsilon^{knm}\,. \tag{2.24}$$

This was the case considered in [1]. The configuration is a non-commutative two sphere describing D0 branes polarized by the $\tilde{N}$ field.

[7]The ε tensor is written in flat background; i.e. we accord no significance to the location of the indices on it.

[8]Without loss of generality, we can assume that this constant of proportionality is positive, i.e. the metric g_{ij} is positive definite; it is also non-singular, and symmetric. Given that we are mapping spacetime indices to indices in group space, this corresponds to our freedom to align the orientations of the two frames with respect to each other.

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Next, consider the case where $\tilde{N} = 0$ and M_{ij} is non-trivial. Equation (2.23) is then a simple quadratic matrix equation. The coordinate system we have chosen, the inertial frame at point P, leaves a remnant of spacetime coordinate invariance at our disposal. We can use an SO(3) subset of this gauge freedom to diagonalize M_{ij}. We write $M_{ij} = \mathrm{diag}(M_1, M_2, M_3)$ and define

$$a_1 \equiv (M_1 - M_2 - M_3)^{1/2} \, , \quad a_2 \equiv (M_2 - M_1 - M_3)^{1/2} \, , \quad a_3 \equiv (M_3 - M_2 - M_1)^{1/2} \, . \tag{2.25}$$

The only symmetry of equation (2.23) is the group S_3 permuting the eigenvalues of M_{ij}. In this coordinate system, the only solution is then given by $g_{ij} = \mathrm{diag}(g_1, g_2, g_3)$ with

$$g_1 = +\frac{a_2 a_3}{a_1} \, , \qquad g_2 = +\frac{a_1 a_3}{a_2} \, , \qquad g_3 = +\frac{a_2 a_1}{a_3} \, . \tag{2.26}$$

We also have the solution with negative g_i's, corresponding to the physically equivalent situation $\Phi^i \to -\Phi^i$ (see footnote 9 above). We need to require

$$a_i^2 \geq 0 \qquad \forall \, i \, . \tag{2.27}$$

Otherwise, the Cartan-Killing metric is complex, and the matrices Φ^i must be *anti*-hermitian; i.e. the coordinates of the D0 branes would be complex. We may consider the case where any two of the a_i^2's are negative; however, this brings one of the metric eigenvalues onto the other branch of the square root; the signature of the Cartan metric changes and the corresponding group is not compact; it cannot be embedded in a compact SU(N). The parameter space available to us by condition (2.27) is depicted in figure 1(a).

The allowed region is where all the eigenvalues of M_{ij} are non-positive and lie within the shaded pyramid. The normalization between the canonical form of the SU(2) generators $[\tau^i, \tau^j] = 2 \, i \, \varepsilon_{ijk} \tau^k$ and ours is

$$\Phi^i = \frac{a_i}{2} \tau^i \, . \tag{2.28}$$

The D0 branes form a non-commutative ellipsoid encoding the content of the background field M_{ij} in the sizes of the radii. The sides of the pyramid in figure 1(a) are singular planes where one of the a_i's vanishes and the ellipsoid gets squashed into a disk. We have a singular solution also when one of the eigenvalues of M_{ij} is zero; then the other two must be equal and two of the a_i's vanish; we are then at one of the edges of the pyramid. The ellipsoid has collapsed in this scenario into an infinitely thin cigar. The solution fails when two or more eigenvalues of the background M_{ij} matrix vanish. In backgrounds that explore the parameter space beyond the shaded pyramid, we need to look for structurally different solutions; we will come back to this issue in section 4.

We also should look at the potential energy of this configuration, to ascertain that these are energetically favored over the trivial configuration. We will need the

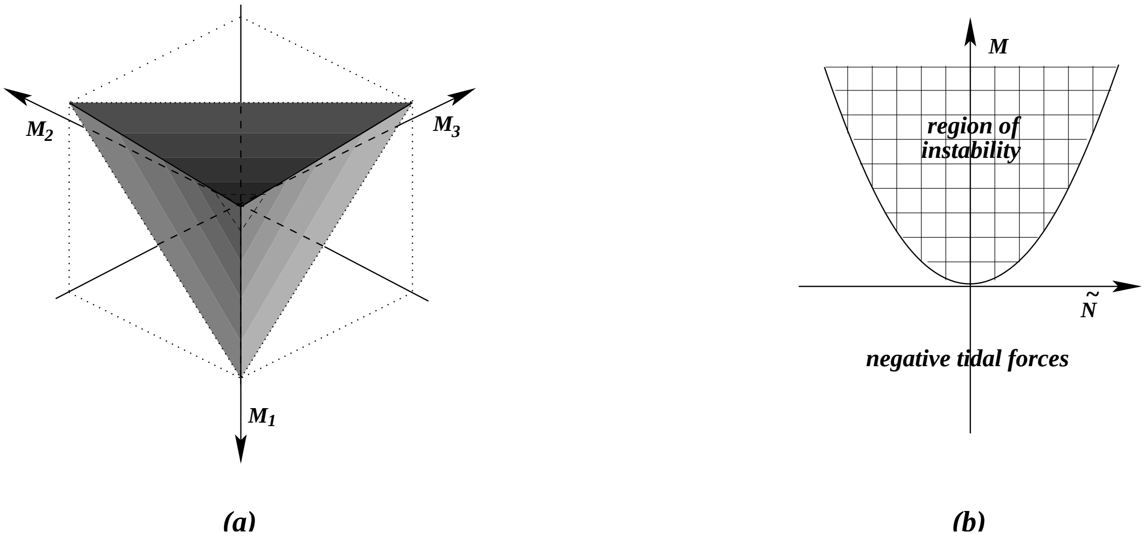

Figure 1: (a) The parameter space of the non-commutative ellipsoid. The region within the pyramid is associated with polarized states. The apex of the pyramid is at the origin and the region depicted is the negative quadrant; (b) The phase structure of the D0 brane vacuum configurations in the presence of an isotropic M_{ij} and $\tilde{N}$ background fields. Everywhere outside the shaded region, the g_+ solution of (2.42) prevails.

trace of the σ^i matrices, which can be easily found using the Wigner-Eckart theorem[9]

$$h^{ij} \equiv \mathrm{Tr}\left\{\Phi^i \Phi^j\right\} = a_i a_j \frac{(N^2-1)N}{12} \delta^{ij} \qquad \text{(no sum over } i, j) \,. \tag{2.30}$$

We write the potential energy in the general case when $\tilde{N} \neq 0$ in a suggestive form

$$V = \frac{1}{g_{\mathrm{str}} \ell_{\mathrm{str}}} \lambda^2 h^{ij} \left(\tilde{N} g_{ij} + g_{ik} g_{kj} - \frac{1}{2} g_{kk} g_{ij} \right) , \tag{2.31}$$

where we have used equation (2.23). For the case at hand, with $\tilde{N} = 0$ and h^{ij} given by (2.30), we get

$$V = -\frac{1}{g_{\mathrm{str}} \ell_{\mathrm{str}}} \frac{\lambda^2}{24} N(N^2-1) \; (a_1 a_2 a_3) \; g_{ii} \,. \tag{2.32}$$

The potential energy for the configuration where all the D0 branes sit on top of each other at $\Phi^i = 0$ is zero. To claim a preferred or competing configuration, we need that the energy given by (2.32) be non-positive. The M_i's are non-positive, while the metric g_{ij} and the a_i's are all positive definite. We then always have $V < 0$.

[9] The Cartan-Killing metric is given by

$$k^{ij} = 2 \; a_1 a_2 a_3 \; g^{ij} = C^{ik}{}_l C^{jl}{}_k \,. \tag{2.29}$$

There are also configurations where all of the Φ^i's are in the Cartan subalgebra of SU(N). These correspond to non-static scenarios that can boundlessly lower the energy. We will address their relevance in the dynamics and to the stability of our solutions in [38].

2.3 Analyzing the ellipsoid

Physically, in the region of the parameter space where the various conditions outlined above are satisfied, we are describing a puffed ellipsoid of D0 branes, with the relative sizes of the three radii of the ellipsoid being related to the anisotropy in M_{ij}. Evaluating the components of the Riemann tensor in the local inertial coordinate system at point P, we have

$$R_{i0j}{}^{0} = \frac{1}{2} G_{00,ij} = R_{ij} - R_{ikj}{}^{k} \equiv \tau_{ij} - R_{ikj}{}^{k} \,. \tag{2.33}$$

M_{ij} can then be also thought of as encoding information about the curvature of the transverse space (the second term) and the spatial part of the energy momentum tensor hidden in what we call τ_{ij}. We can use the supergravity equations of motion to write M_{ij} explicitely in terms of the field components and derivatives in the three dimensional transverse space

$$\begin{aligned} M_{ij} = \frac{1}{2}\left(\phi_{,i}\phi_{,j} + \phi_{,ij} + R_{ikj}{}^{k}\right) - \frac{1}{8} H_{i\mu\nu} H_{j}{}^{\mu\nu} - \frac{1}{4} e^{2\phi}\left(F^{(2)}_{i\mu} F^{(2)\mu}_{j} - \frac{1}{4}\delta_{ij}\left(F^{(2)}\right)^2\right) - \\ -\frac{1}{24} e^{2\phi}\left(\tilde{F}^{(4)}_{i\alpha\beta\gamma} \tilde{F}^{(4)\alpha\beta\gamma}_{j} - \frac{1}{8}\delta_{ij}\left(\tilde{F}^{(4)}\right)^2\right) - \frac{1}{2} F^{(2)}_{0i,j} \,, \end{aligned} \tag{2.34}$$

with $\tilde{F}^{(4)} \equiv F^{(4)} - C^{(1)} \wedge H^{(3)}$. We emphasize that the dynamics of D0 brane polarization is consistent with the equations of motion satisfied by the background fields. We had originally hoped that we may find through the worldvolume dynamics conditions on the background fields that lead to the supergravity equations themselves. However, we find instead that information from the worldvolume theory as extracted through this approach is insufficient. There exist infinite classes of solutions to the background fields equations that can polarize D0 branes.[10]

If we consider the dynamics of two nearby geodesics at point P, their relative acceleration is related to the separation z^i between them by the well known equation

$$a^i = R_{j0i}{}^{0} z^j = -2 M_{ij} z^j \,. \tag{2.35}$$

The condition that the eigenvalues of the matrix M_{ij} must be negative is simply the statement that the space must be curved such that two nearby geodesics repel each other. The polarization phenomena has to do with the effect of tidal-like forces in M_{ij} expanding the N D0 branes against the binding forces due to strings stretched

[10] A particular example is the background geometry of near extremal D6 branes.

between them. There is apparently a scenario where a balance between these two competing effects is possible and one finds a non-commutative ellipsoid, encoding data about the conventional gravitational tidal forces, second derivatives in the dilaton, and the gradient of the background D0 brane electric field (see equation (2.11)). Note however that there is an issue of stability, alluded to above, with regards to this statement. As presented, these solutions are at the extrema of the potential, but it is unclear whether they would want to stay there. In particular, there are directions about the $\Phi^i = 0$ point within the Cartan subalgebra of $\mathrm{SU}(N)$ that will drive the energy to arbitrarily small values.[11] These modes may destabilize the ellipsoid as well. A careful analysis is needed to ascertain this issue. Even if there exists a stabilization mechanism about the ellipsoidal configuration, the presence of a dip in the potential energy about $\Phi^i = 0$ suggest that these configurations may not be reached in a generic scenario. We will attempt to elaborate on all of these problems in an upcoming work [38]. For now, looking at our original action, it is easy to read off the time scale for possible decay. It is set by the typical wavelength of the background fields. We will see below that this implies that smaller configurations are longer lived. It then becomes important to determine the relevant statement with respect to the time scale useful for observation.

It is also worth emphasizing that the response of the D0 brane configuration to these tidal forces is somewhat exotic. As depicted in figure 1(a), there are regions of the parameter space where all the M_i's are negative, yet our solution fails. Too much anisotropy in the background space is destructive. For example, when $M_1 = M_2 + M_3$, with all M_i's negative, the ellipsoid collapses in one direction to become a disk. This phenomenon is a probe into the internal distributions of the forces amongst the D0 branes, including interactions resulting from strings stretched between them.

A good measure of the extent the D0 branes spread out was introduced by [21]. We denote the size of the configuration by r^2 and define

$$\frac{r^2}{\lambda^2} \equiv \frac{\mathrm{Tr}\left\{\Phi^i \Phi^i\right\}}{N} = -\frac{N^2-1}{12} M_{ii} \,. \tag{2.36}$$

Looking back at equation (2.1), it is easy to see that the regime of validity for the expansion in λ can be stated as the condition

$$\lambda^2\, \mathrm{Tr}\left\{\Phi^i \Phi^j\right\} \partial_i \partial_j \ll N \,. \tag{2.37}$$

If we write L for the characteristic length scale over which the fields vary, this implies

$$r^2 \ll L^2 \,. \tag{2.38}$$

The size of the polarized configuration must be much less than the characteristic length scale over which the background fields vary. Note that this was also the

[11] This issue was pointed out to me by A. Volovich and M. Spradlin.

condition needed so that the trajectory of the center of mass of the D0 branes is not affected by the details of its polarized shape. Using equation (2.36) in (2.38), we get the relation

$$N \ll \mathcal{A} \equiv \frac{L^2}{\alpha'}\,, \tag{2.39}$$

where $\mathcal{A}$ is the area constructed using L, in string units.[12] Note that we also need that $L \gg \ell_{\mathrm{str}}$. As the bound in equation (2.39) is approached, our description of the problem breaks down. It is a bound on the number of degrees of freedom, the D0 branes, that can be placed at point and consistently be described through the DBI action. More on this in the Discussion section.

The size of the configuration as given in (2.36) is

$$r \sim \ell_{\mathrm{str}} N \left(\frac{\ell_{\mathrm{str}}}{L}\right)\,. \tag{2.40}$$

The larger the background length scale L, the smaller the size of the configuration; and the size grows with the number of D0 branes. This may be viewed as a local manifestation of the UV-IR relation [12]; simply put, the "dispersion relation" of our probes in the curved background. And for $N < L/\ell_{\mathrm{str}}$, where L/ℓ_{str} is always much greater than one, the size of the configuration is substringy, yet we are in a valid regime for the computation. As is well known, the D0 branes are naturally good probes of Planck scale distances in space.

The physical data encoded in M_{ij} consisted of three numbers, and a Lie algebraic structure is enough to resolve this information, transfering it into the shape of the non-commutative ellipsoid. Physically, the matrix M_{ij} gets encoded in a selective set of stretched open strings. We will see in section 3 that, to resolve more structure of the background space, such as higher derivatives of the metric, more "links" between the D0 brane would be needed, and a Lie algebraic structure is not enough.

2.4 A non-commutative sphere and balance of forces

In this section, we consider solutions to (2.23) with both $\tilde{N}$ and M_{ij} non-zero. Solving the full anisotropic case involves as much pleasure as solving Maxwell's equations in a cavity of arbitrary shape, void of any symmetries. Most of the dynamics of the competition between the various forces can be demonstrated by considering an isotropic configuration, where the eigenvalues of the matrix M_{ij} are all equal $M_1 = M_2 = M_3 \equiv M$; consequently, the solution is a non-commutative sphere [22, 21] and we write the eigenvalues of g_{ij} as $g_1 = g_2 = g_3 \equiv g$. We then have the equation

$$g^2 - 3\tilde{N} g + M = 0\,, \tag{2.41}$$

[12] We remind the reader that this is a statement in the string frame; the corresponding equation in the conventional Einstein frame is with respect to area in Planck units.

with solutions

$$g_{\pm} = \frac{3\tilde{N}}{2}\left(1 \pm \left(1 - \frac{4M}{9\tilde{N}^2}\right)^{1/2}\right), \tag{2.42}$$

with the condition

$$M \leq \frac{9}{4}\tilde{N}^2 . \tag{2.43}$$

We learned from our previous discussion in section 2.2 that the effect of $\tilde{N}$ is to create repulsive forces between the D0 branes. This means we may now expect solutions with $M > 0$ as well, where equation (2.43) becomes relevant.

To decide between the two solutions in (2.42), we evaluate the potential energy of the configuration

$$V = \frac{3}{g_{\rm str}\ell_{\rm str}}\lambda^2 \frac{N(N^2-1)}{24} g_{\pm}^3 \left(2\tilde{N} - g_{\pm}\right) . \tag{2.44}$$

And requiring $V \leq 0$, we arrive at the condition

$$\left(1 \pm \left(1 - \frac{4M}{9\tilde{N}^2}\right)^{1/2}\right)\left(1 \mp 3\left(1 - \frac{4M}{9\tilde{N}^2}\right)^{1/2}\right) \leq 0 . \tag{2.45}$$

We then can easily determine the following possibilities: for $M > 0$, the only solution is given by g_+, subject to the condition

$$M < 2\tilde{N}^2 , \tag{2.46}$$

which is stronger than (2.43), and hence prevails. For $M < 0$, equation (2.43) is satisfied, and we have both solutions $g_{\pm}$ being possible. The one that prevails is the one with lower energy. Looking at (2.42), we see that we have $|g_+| > |g_-|$. Hence, if

$$g_+\left(2\tilde{N} - g_+\right) < g_-\left(2\tilde{N} - g_-\right) \Rightarrow V_+ < V_- . \tag{2.47}$$

Rearranging this equation, and using (2.42), we find that the energetically favored solution for $M < 0$ is g_+ again. We arrive at the simple phase structure shown in figure 1(b). In the shaded region, our solutions are disfavored even classically. We see that the interactions appearing in (2.1) may predict interesting phase structures of vacuum configurations of D brane probes; one that results from the competition of the various couplings to the background fields.

3. Beyond Lie algebras

It is apparent from our discussion in the previous sections that the Lie algebraic structure was exhausted in the process of encoding the space-time data into the D0 brane matrices. As more details and moments get probed by higher order couplings, it is then natural to ask how does this additional information get stored in the Φ^i matrices. In this section, we explore terms in (2.1) cubic in λ, by outlining a formal recursive prescription, involving an expansion in λ, which allows us to go beyond the Lie algebraic structure. We again confine our discussion to a three dimensional subspace of the transverse space, even though many of the equations we will write are relevant to more general cases as well.

The equations of motion we consider are a specialization of (2.17)

$$2\Phi^i M_{ni} + \left[\Phi^i, \left[\Phi^i, \Phi^n\right]\right] + \frac{3}{2}\lambda\left\{\Phi^i, \Phi^j\right\}T_{nij} = 0\,, \tag{3.1}$$

with M_{ij} and T_{ijk} totally symmetric tensors. From the structure of (3.1), we see that the Lie algebra

$$\left[\Phi^i, \Phi^j\right] = C^{ij}{}_k \Phi^k \tag{3.2}$$

would account for the term involving M_{ij} as described in the previous sections. But then, the term involving the anti-commutator would introduce matrices outside the space spanned by the original Φ^i's; we have

$$\left\{\Phi^i, \Phi^j\right\} = 2h^{ij}\mathbf{1} + \sigma^{ij}\,, \tag{3.3}$$

where

$$h^{ij} \equiv \mathrm{Tr}\left\{\Phi^i\Phi^j\right\}\,. \tag{3.4}$$

Generically, the six $N \times N$ matrices σ^{ij} are not in the SU(2) algebra. Note however that, in the case $N = 2$, the Pauli matrices anticommute such that the σ^{ij}'s are zero. These matrices arise as we add more and more D0 branes, increasing the size of the matrices. Physically, this is an interesting statement. With a few D0 branes, we can resolve so much of the spacetime structure. As we introduce higher derivative perturbations in the background fields, polarized configurations with fewer D0 branes will tend to become unstable sooner than configurations with larger number of D0 branes. This is because, to resolve additional structure in the background fields, one adds more stretched strings or "links" between the D0 branes; and this pool of matrix data is limited by the size of the matrices, the N^2 entries in the Φ^i's. This is a generic idea; an identical phenomenon can be visualized for example in electromagnetism with respect to the competition between the number of charges and higher moments of electric field backgrounds. We assume that N is large enough so that a solution to the problem at hand is possible. A naive counting suggest a number greater than 4. The idea involved in solving (3.1) is to "add additional links" between the D0 branes by turning on off-diagonal elements in the D0 brane matrices, so as to balance the shearing forces due to the background field T_{nij}.

Consider a solution of the form

$$\Psi^i = \Phi^i + \lambda e^i_{kl}\sigma^{kl} + \lambda\gamma^i\mathbf{1} + O(\lambda^2)\,. \tag{3.5}$$

The Φ^i's satisfy the SU(2) algebra. To order λ, we are perturbing by the traceless hermitian matrices σ^{ij} appearing on the right side of (3.3). These contain the additional "links" that need to be activated between the D0 branes. The c-number parameters e^i_{kl} play the role of vielbeins; they connect the space time index to the matrix space spanned by the σ^{ij}'s. Looking at equation (3.1), we see that adding the σ^{ij}'s is not enough due to the term proportional to the identity in (3.3). This requires adding the term involving γ^i in (3.5). Physically, this is the statement that some of the higher moments in T_{nij} shift the center of mass of the configuration; these are the modes given by $T_{nij}h^{ij}$. The presence of the identity in (3.5) is needed to account for effects of the size of the configuration on the trajectory of the center of mass. The U(1) and SU(N) in U(N) do not decouple at this order, and this is incorporated in the trace part of T_{nij}. The resulting configuration is shifted off-center.

We now can easily compute

$$\left[\Phi^k, \sigma^{ij}\right] = C^{ki}{}_{l}\sigma^{lj} + C^{kj}{}_{l}\sigma^{il}\,, \tag{3.6}$$

where we have used the fact that

$$C^{ijk} \sim C^{ij}{}_{l}h^{lk} \tag{3.7}$$

is total antisymmetric on i, j, k, given that h^{ij} is propotional to the Cartan-Killing metric of SU(2). Equation (3.6) is an important ingredient of our prescription. It is related to a well known identity and corresponds to the statement that the commutator of Φ^i with the σ^{kl}'s closes onto the space spanned by the σ^{kl}'s. We compute the commutator of our ansatz (3.5)

$$\begin{aligned}\left[\Psi^i, \Psi^n\right] &= C^{in}{}_{m}\Phi^m + 2\lambda\left(e^n_{kl}C^{il}{}_{m} - e^i_{kl}C^{nl}{}_{m}\right)\sigma^{km} + O(\lambda^2)\\ &= C^{in}{}_{k}\Psi^k + 2\lambda D^{in}_{ml}\sigma^{ml} - \mathbf{1}\lambda C^{in}{}_{k}\gamma^k + O(\lambda^2)\\ &= C^{in}{}_{k}\Psi^k - \lambda C^{in}{}_{k}\gamma^k\mathbf{1} - 4\lambda D^{in}_{ml}h^{ml}\mathbf{1} + 2\lambda D^{in}_{ml}\left\{\Psi^m, \Psi^l\right\} + O(\lambda^2)\,,\end{aligned} \tag{3.8}$$

where we have introduced

$$D^{ij}_{ml} \equiv e^j_{kl}C^{ik}{}_{m} - e^i_{kl}C^{jk}{}_{m} - \frac{1}{2}e^k_{lm}C^{ij}{}_{k}\,. \tag{3.9}$$

These D-structure constants are symmetric in the lower indices, and antisymmetric in the upper. We substitute all these relations into (3.1) and obtain the following equations:

- To $O(\lambda)$, we have terms involing only the Φ^i as before. We get equations that determine the structure constants $C^{ij}{}_{k}$ in terms of M_{ij}

$$2M_{nm} + C^{in}{}_{l}C^{il}{}_{m} = 0\,. \tag{3.10}$$

- To $O(\lambda^2)$, the trace part (3.1) determines the γ^i's

$$3T_{nij}h^{ij} + 2M_{ni}\gamma^i = 0\,. \tag{3.11}$$

- To $O(\lambda^2)$, we have terms involving the σ^{ij}'s; and we find equations that determine the D^{kl}_{ij} in terms of T_{nij}

$$\frac{3}{4}T_{nlj} + C^{in}{}_k D^{ik}_{lj} + 2C^{im}{}_j D^{in}_{lm} = 0\,. \tag{3.12}$$

  In deriving this equation, we have made use of equation (3.10) that appears at a lower order in λ. Note that the vielbeins do not appear explicitely, and we end up solving for the D-structure constants. Formally, from these, we can detemine the veilbeins e^{kl}_i. Naively, it also appears we have more unknowns than equations, and there is an issue about the uniqueness of solutions to (3.12). Such freedom may correspond to symmetries, or some of this information may be needed when other fields get turned on.

The generalized algebra in (3.8) is an extension of the Lie algebra defined by the $O(\lambda^2)$ solution. It has structure similar to a q-deformed algebra [13, 14]. It is instructive to briefly comment on how this structure is realized explicitely in matrix space. The SU(2) part of the algebra in an N dimensional representation correspond to matrices of the form, schematically

$$\begin{pmatrix} x & x & 0 & 0 & 0 & \\ x & x & x & 0 & 0 & \\ 0 & x & x & x & 0 & \cdots \\ 0 & 0 & x & x & x & \\ 0 & 0 & 0 & x & x & \\ & & \vdots & & & \end{pmatrix} \tag{3.13}$$

The corresponding σ^{ij}'s by which we perturb the solution span matrices of the form

$$\lambda \begin{pmatrix} x & x & x & 0 & 0 & \\ x & x & x & x & 0 & \\ x & x & x & x & x & \cdots \\ 0 & x & x & x & x & \\ 0 & 0 & x & x & x & \\ & & \vdots & & & \end{pmatrix} \tag{3.14}$$

We see that the additional data is encoded in "next to nearest neighbour" links between the D0 branes. Perhaps there is a general pattern where we explore the matrix space starting along the diagonal and moving outward as we expand to higher orders in λ.

4. A non-compact solution

In this section, we come back to (2.14), with the field A_i set to zero

$$2\Phi^i M_i = \left[\Phi^i, \left[\Phi^n, \Phi^i\right]\right], \tag{4.1}$$

looking for solutions beyond the pyramid defined by figure 1(a). We consider the case where one eigenvalue of M_{ij} is zero, and the other two *positive* and equal. It appears from our previous discussion that a compact configuration is not possible. We will therefore look for non-compact solutions, such as the one that solves the flat space case $[\Phi^1, \Phi^2] = i\theta\mathbf{1}$ considered in [23, 9]. These vacua will necessarily have infinite energy but may be stabilized by appropriate boundary conditions at infinity. We may also want to consider the Matrix theory limit $g_{\rm str}, \ell_{\rm str} \to 0$ with $\ell_{\rm str}/g_{\rm str}^{1/3}$ fixed, so that terms of order λ^3 and beyond in the action can be ignored all the way to asymptotic infinity in spacetime.

Working in a diagonal basis of the M_{ij} matrix, the matrix entries are chosen as

$$M_1 = 0\,, \qquad M_2 = M_3 = M > 0\,. \tag{4.2}$$

Inspecting equation (4.1), we can easily write the solution as the closed algebra

$$\left[\Phi^2, \Phi^3\right] = i\theta\mathbf{1}\,, \qquad \left[\Phi^1, \Phi^3\right] = -i\sqrt{M}\Phi^2\,, \qquad \left[\Phi^1, \Phi^2\right] = -i\sqrt{M}\Phi^3\,; \tag{4.3}$$

and we can represent Φ^1 as

$$\Phi^1 = \frac{\sqrt{M}}{2\theta}\left((\Phi^3)^2 - (\Phi^2)^2\right)\,. \tag{4.4}$$

This seems to describe a non-commutative hyperboloid; a microscopic description of a deformed D2 brane extending all the way to infinity. Note that, to realize this algebra, we need to take matrices of infinite size $N \to \infty$.

The potential energy of this configuration is given by

$$V = \frac{\lambda^2}{g_{\rm str}\ell_{\rm str}}\left(M\,{\rm Tr}\left\{(\Phi^2)^2 + (\Phi^3)^2\right\} + \frac{\theta^2}{2}N\right)\,. \tag{4.5}$$

The second term is the same for the plane configuration in flat space as well [23, 9] and we may choose to substract it to quantify the energy content of our configuration. We can evaluate the trace over the Heisenberg operators by "regulating" the divergence with N

$${\rm Tr}\,q^2 = \frac{\theta}{2}\sum_{j=0}^{N-1} 2j+1 = \frac{\theta}{2}N^2\,. \tag{4.6}$$

We then consider the excess energy

$$\frac{V}{N^2} \to \frac{3}{4}\frac{1}{g_{\rm str}\ell_{\rm str}}(\lambda\theta)(\lambda M) \sim \frac{1}{g_{\rm str}\ell_{\rm str}}\frac{\Theta}{(L/\ell_{\rm str})^2}\,. \tag{4.7}$$

Θ is the scale of non-commutativity on the hyperboloid in string units. This excess energy seems to be distributed amongst all N^2 entries of the matrices.

In general, we may expect exotic solutions such as this in regions of the background field parameter space where compact configurations cannot exists. We may expect that we are allowed to use contractions and deformations of SU(2), as long as we end up with a consistant, closed algebras that is energetically or dynamically favoured.

5. Discussion

Two statements from our analysis of the non-commutative ellipsoid are worth further elaboration. Equation (2.39) determines the regime of validity of the DBI expansion. It places a bound on the number of D0 branes in terms of the length scale over which the background fields are varying. It is a statement about limiting one bit of information per Planck area, with the measure of area being defined in a local manner, using the characteristic scales of the background.[13] As we tune the wavelength L of background fields to smaller values, the configuration of D0 branes expands in size, until matching the characteristic size of the background field variations. At this point, L is small enough that we have one D0 per Planck area. Our analysis is about D0 branes acting as probes to the structure of spacetime, ignoring back reaction effects due to the D0 branes themselves. As this bound gets saturated, our formalism breaks down, and the situation needs to be described within the context of the full string theory. We may expect that back reaction effects will become important before reaching the critical point, and equation (2.39) should be interpreted as an approximate scaling relation. Note however that, at the saturation point, background curvature scales can be very small for large values of N, well within the supergravity approximation regime. In view of independent observations about Holographic bounds and black hole entropy, we may interpret (2.39) as more that just a statement restricting the regime of validity of an approximation scheme; but one that is rooted in fundamental physics.

The second interesting point has to do with a local formulation of the UV-IR correspondence. Equation (2.40) shows that the D0 brane ellipsoid shrinks in size with larger wavelengths of the background space. Let us imagine perturbing the vacuum ellipsoidal configuration so as to write the theory of surface fluctuations as was done in [8, 9, 25]. We can represent the matrix algebra over the space of smooth functions by introducing the appriopriate star product [7, 8, 9, 24]. We would be describing the microscopic dynamics of a D2 brane-like object contructed from D0

[13]Presumably, the statement is sensitive to the fact that we restricted the dynamics to a three dimensional subset of the transverse space and that the resulting object is a D2 brane; hence the quadratic power in L. It is an interesting problem to understand the same issue in higher dimensions.

branes. The 2+1 dimensional worldvolume theory is non-commutative and lives on a compact space. The size of the configuration is given by (2.40); we remind the reader that the metric of relevance is (2.5), i.e. it is flat. Hence, the IR cutoff in this theory is

$$\Sigma \sim r \sim \ell_{\rm str} \frac{N}{L/\ell_{\rm str}} \,. \tag{5.1}$$

In [8, 9], an interesting general statement was made relating the IR cutoff Σ, the non-commutativity parameter θ and the number of D0 branes that underly a non-commutative field theory

$$N \sim \frac{\Sigma^2}{\theta} \,. \tag{5.2}$$

The non-commutativity scale θ plays the role of UV regulator; equation (5.2) states that the number of D0 branes underlying the non-commutative dynamics is given by the ratio of the IR to the UV cutoff; this is a simple yet important statement of a general character. We can use it to estimate the scale of non-commutativity in the worldvolume theory resulting from perturbing the ellipsoidal configuration. Using (5.1) as the IR cutoff, we find (in the string frame)

$$\sqrt{\theta} \sim \sqrt{N} \frac{\alpha'}{L} \,. \tag{5.3}$$

The larger the wavelengths in the background space, the smaller the length scale of non-commutativity in the world-volume theory. This is a local manifestation of the usual UV-IR correspondence [26, 12].

Restricting the discussion to a three dimensional subspace led to configurations which are microscopic realizations of D2 branes [27, 28]. One should take note of the fact that these polarized states resulted from tidal-like forces, without the need to turn on the RR gauge fields corresponding to a higher dimensional brane. The end result will carry D2 brane dipole moment as dictated by the couplings appearing in (2.1). Different parts of the droplets of D0 branes "fall" with different accelerations; hence the D0 branes expand while retaining a coherent shape. This effect is achieved by gravitational tidal forces, gradients in the background D0 brane one form gauge field, and non-zero second derivatives of the dilaton field.

The stability of the configurations we found is an issue that needs to be determined. This can be done by perturbing the solutions and looking for tachyonic modes. And in case of decay, the time scale is important to determine the physical relevance of these configurations. As presented, one should view our solutions as extrema of the action with interesting properties that probe the issue on how to distribute smooth spacetime information into matrices. We tackle the stability problem in detail in [38].[14]

[14] In particular, in [38], we find that these ellisoidal configurations are unstable; they correspond to saddle points of the potential. The timescale associated with these saddle points is however large for large numbers of D0 branes and small enough string coupling.

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On a few technical notes, it is worthwhile pointing out that, in expanding (2.1), some simplifications arose due to the symmetrized trace prescription introduced in [19]. It is known that this procedure fails at higher orders [30], but none of our calculations probed this regime. Another aspect has to do with representing the non-commutative algebras of matrices on the space of functions. It may be intructive to study the worldvolume theories on the polarized configurations, as a prelude to connecting to a smooth macroscopic D2 brane picture. The star operator will arise in this context, and the mathematical foundations of this, for both Lie and q-deformed algebras, have recently been explored in [13, 14, 29, 31, 32]. In particular, there is a well-defined star product associated with the D-structure constants introduced in (3.9).

There are several immediate extensions of these ideas that are of interest. For one, it would be useful to understand similar polarization phenomena in higher than three dimensions. We should expect for example D4 or D6 branes arising from the polarizing effects of tidal forces. The algebras of relevance maybe $SU(2)\times SU(2)$ and $SU(3)$. The eight generators of the latter minus its two Cassimirs perhaps provide the appropriate embedding of the worlvolume in the spacetime. Given the subtleties associated with the five brane, such an approach may be too naive after all [1].

It would be interesting to understand the pattern of higher order effects in the DBI action in the structure of vacuum solutions. The question is about the forms of generalized algebras that can arise in encoding information about spacetime fields into matrices as an expansion in the moments of the fields. It would also be helpful to write realizations of these generalized algebras in explicit examples, to develop intuition about the dynamics involved.

In [33], the Chern-Simmons term was generalized to include the effects of additional couplings to the spacetime curvature. The effects of these need to be considered in a consistent analysis specially when background curvature scales are a large. They would introduce couplings with even powers of the Ricci tensor and the RR gauge fields. Another interesting issue is to consider dynamical situations; such as when a gravitational wave sweeps past a configuration of D0 branes. This seting will explore a new set of interaction terms arising in (2.1) which we did not consider.

Finally, an important issue may be to understand this polarization phenomena in the presence of time-space non-commutativity [34, 35, 36]. The relevant setup may be to study D-instanton dynamics [37] near black hole horizons. The flipping of the light-cone at a horizon may necessitate consideration of non-commutation of time and space. We hope to report on this issue in the future.

A. A few technical details

In this appendix, we sketch some of the computational details of the main text. The main ingredients in expanding (2.1) to order λ^3 under the conditions outlined in section 2.1 are the expansion of the supergravity field as in (2.4), and the expansion

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of det Q. The term containing $Q^{-1} - \delta$ of (2.1) never arises in our discussion, and the pull-back is trivial in the static scenario. The determinant of the Q matrix expands to

$$
\begin{aligned}
(\det Q)^{1/2} = 1 &+ i\frac{\lambda^2}{2}\left[\Phi^i, \Phi^k\right]\Phi^l B_{ki,l} - \frac{\lambda^2}{4}\left[\Phi^i, \Phi^j\right]\left[\Phi^i, \Phi^j\right] + \\
&+ i\frac{\lambda^3}{4}\left[\Phi^i, \Phi^k\right]\Phi^m \Phi^l B_{ki,ml} + \frac{\lambda^3}{2}\left[\Phi^i, \Phi^j\right]\left[\Phi^j, \Phi^m\right]\Phi^l B_{mi,l} - \\
&- i\frac{\lambda^3}{6}\left[\Phi^i, \Phi^k\right]\left[\Phi^k, \Phi^l\right]\left[\Phi^l, \Phi^i\right] + O(\lambda^4)\,. \qquad \text{(A.1)}
\end{aligned}
$$

The last two terms do not contribute; the first of these vanishes due to the symmetry in the i and m indices using the symmetrized trace prescription; the second vanishes also because it arises in the symmetrized trace.

The Chern-Simmons terms expand to

$$
\begin{aligned}
S_{CS} = \frac{1}{g_{\rm str}\ell_{\rm str}} \int {\rm STr}\Big\{ P\Big[& C^{(1)} + i\lambda i_\Phi i_\Phi C^{(3)} + i\lambda^2 i_\Phi i_\Phi \left(C^{(1)} B, i\right)\Phi^i - \\
& - \frac{\lambda^2}{2}\left(i_\Phi i_\Phi\right)^2 C^{(5)} + O(\lambda^3)\Big]\Big\}\,. \qquad \text{(A.2)}
\end{aligned}
$$

with

$$
\begin{aligned}
P\left[C^{(1)}\right] &= C_0^{(1)} + \lambda \Phi^i C_{0,i}^{(1)} + \frac{\lambda^2}{4}\left\{\Phi^i, \Phi^j\right\} C_{0,ij}^{(1)} + O(\lambda^3)\,; \\
i\lambda P\left[i_\Phi i_\Phi\left(C^{(3)} + C^{(1)} B\right)\right] &= i\frac{\lambda}{2}\left[\Phi^j, \Phi^i\right]\left(C_{ij0}^{(3)} + \lambda \Phi^k C_{ij0,k}^{(3)}\right) + i\frac{\lambda^2}{2}\left[\Phi^j, \Phi^i\right]\Phi^k C_0^{(1)} B_{ij,k}\,; \\
-\frac{\lambda^2}{2} P\left[\left(i_\Phi i_\Phi\right)^2 C^{(5)}\right] &= -\frac{\lambda^2}{8}\left[\Phi^l, \Phi^k\right]\left[\Phi^j, \Phi^i\right] C_{ijkl0}^{(5)}\,. \qquad \text{(A.3)}
\end{aligned}
$$

Puting everything together, we get the actions given in (2.9) and (2.15).

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